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ORIGINAL RESEARCH

# Optimality conditions and duality results for non-differentiable interval optimization problems

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**Abstract** In present study, an interval optimization problem is addressed in which both objective and constraint functions are non-differentiable. The existence of the solution for this problem is investigated. Further, the necessary and sufficient optimality conditions are explored. Moreover, the weak and strong duality relations between the primal and the corresponding dual interval optimization problem are established. Counterexamples are discussed to justify the present work.

**Keywords** Interval optimization problem · Non-differential optimization problem · Efficient solution · Convex function

**Mathematics Subject Classification** 90C25 · 90C29 · 90C30 · 90C46

## 1 Introduction

Parameters in the objective and constraint functions of a general optimization problem have been considered as fixed real numbers. However, in most of the real life situations, these parameters may contain some uncertainty due to the presence of indistinct information in the data set. Such type of uncertainty can be easily interpreted in terms of closed intervals. Therefore, the optimization problem with parameters as closed intervals is called an interval optimization problem ($IOP$). The optimality condi-

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tions and duality theory for $IOP$ plays an important role in optimization theory. The duality relations for linear optimization problem with inexact data were well studied [5, 10, 12–14, 16]. Wu [17–19] and Jayswal et al. [7] discussed duality results and optimality conditions for interval optimization problem in which the objective function is an interval valued, constraint functions are real valued and both are differentiable. Ahmad et al. [1] obtained sufficient optimality conditions and duality results for differentiable interval optimization problems using generalized invexity.

If at least one of the objective and constraint functions in $IOP$ is not differentiable, then $IOP$ becomes a non-differentiable $IOP$. Recently, Sun and Wang [15] gave the concept of LU optimal solution to non-differentiable interval programming problem problem, where objective function is interval valued and constraint functions are real valued. Anurag et al. [2] discussed the optimality criteria using saddle point theory for non-smooth interval optimization problem. The Fritz John and Kuhn-Tucker type necessary and sufficient optimality conditions were described for this problem. They also established desired duality theorems using LU partial ordering. However, the optimality conditions and duality relations for general non-differentiable interval programming problem in which objective and all constraints are interval valued functions, have not been explored so far. Therefore, the aim of the investigation is to study the optimality conditions and duality theory for a general interval optimization problem without assuming the differentiability of the objective as well as constraint functions.

The remaining part of the composition is organized as follows. In Sect. 2, some prerequisites, which are utilized for developing the solution is defined. Existence solution of $IOP$ is discussed in Sect. 3. The necessary and sufficient optimality condition for non-differentiable $IOP$ is described in Sect. 4. In Sect. 5, The dual problem for non-differentiable $IOP$ is discussed and desire duality relations between primal and dual problems are established.

## 2 Preliminaries

The following notations are used throughout the paper :
Bold capital letters denote closed intervals, and small letters denote real numbers.
$I(R)$ is the set of all closed intervals in $R$.
$(I(R))^k$ is the product space $\underbrace{I(R) \times I(R) \times \ldots \times I(R)}_{k \text{ times}}$.
$\mathbf{C}_v^k$ is $k$-dimensional column vector whose elements are intervals. That is,

$$\mathbf{C}_v^k \in (I(R))^k, \mathbf{C}_v^k = (\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_k)^T, \mathbf{C}_j = [c_j^L, c_j^R],\ j \in \Lambda_k, \Lambda_k = \{1, 2, \ldots, k\}.$$

For two real vectors $a = (a_1, a_2, \ldots, a_n)^T$, $b = (b_1, b_2, \ldots, b_n)^T$ in $R^n$, we denote

$$a \geqq_v b \Leftrightarrow a_i \geq b_i;\ a \leqq_v b \Leftrightarrow a_i \leq b_i;\ a >_v b \Leftrightarrow a_i > b_i;$$
$$a <_v b \Leftrightarrow a_i < b_i,\ i \in \Lambda_n.$$

The binary operation $\circledast$ between two intervals $\mathbf{A} = [a^L, a^R]$ and $\mathbf{B} = [b^L, b^R]$ in $I(R)$, denoted by $\mathbf{A} \circledast \mathbf{B}$ is the set $\{a * b : a \in \mathbf{A}, b \in \mathbf{B}\}$, where $* \in \{+, -, \cdot, /\}$ is a binary operation on the set of real numbers. In the case of division, $\mathbf{A} \oslash \mathbf{B}$, it is assumed that $0 \notin \mathbf{B}$. These interval operations can also be expressed in terms of parameters. Any point in $\mathbf{A}$ may be expressed as $a(t) = a^L + t(a^R - a^L),\ t \in [0, 1]$. Algebraic operations of intervals may be explained in parametric form as follows.

$$\mathbf{A} \circledast \mathbf{B} = \left\{a(t_1) * b(t_2) |\ t_1, t_2 \in [0, 1]\right\}. \tag{1}$$

The following prerequisites are required to develop the results in Sects. 4 and 5. The set of intervals $I(R)$ is not a totally order set. Several partial orderings in $I(R)$ exist in the literature. The partial ordering in parametric form is considered due to Bhurjee and Panda [4].

**Definition 1** [4] For $\mathbf{A}, \mathbf{B} \in I(R)$,

$$\mathbf{A} \preceq \mathbf{B} \text{ if } a(t) \le b(t),\ \forall\, t \in [0, 1] \text{ and } \mathbf{A} \prec \mathbf{B} \text{ if } \mathbf{A} \preceq \mathbf{B} \text{ and } \mathbf{A} \ne \mathbf{B}. \tag{2}$$

**Note :** $\mathbf{A} \prec \mathbf{B}$ is equivalent to $a(t) \le b(t), \forall\, t \in [0, 1]$ and for at least one $t' \ne t,\ a(t') < b(t')$.

Many authors [6,9,17] defined interval valued function in several ways. The concept of Bhurjee and Panda [4] for interval valued function is implemented in the parametric form.

**Definition 2** [4] For $c(t) \in \mathbf{C}_v^k$, let $f_{c(t)} : R^n \to R$. For a given interval vector $\mathbf{C}_v^k$, define an interval valued function $\mathbf{F}_{\mathbf{C}_v^k} : R^n \to I(R)$ by

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left\{ f_{c(t)}(x) \;\middle|\; f_{c(t)} : R^n \to R, c(t) \in \mathbf{C}_v^k \right\}.$$

For every fixed $x$, if $f_{c(t)}(x)$ is continuous in $t$ then $\min_{t\in[0,1]^k} f_{c(t)}(x)$ and $\max_{t\in[0,1]^k} f_{c(t)}(x)$ exist. In that case

$$\mathbf{F}_{\mathbf{C}_v^k}(x) = \left[ \min_{t\in[0,1]^k} f_{c(t)}(x),\ \max_{t\in[0,1]^k} f_{c(t)}(x) \right].$$

If $f_{c(t)}(x)$ is monotonically increasing in $t$, then $\mathbf{F}_{\mathbf{C}_v^k}(x) = [f_{c(0)}(x), f_{c(1)}(x)]$.

The convexity property is defined for interval valued function with respect to partial ordering in order to develop the optimality conditions and duality relations for $IOP$.

**Definition 3** [4] Suppose $D \subseteq R^n$ is a convex set. For given $\mathbf{C}_v^k \in (I(R))^k$, the interval valued function $\mathbf{F}_{\mathbf{C}_v^k} : D \to I(R)$ is said to be convex with respect to $\preceq$ if for every $x_1, x_2 \in D$ and $0 \le \lambda \le 1$,

$$\mathbf{F}_{\mathbf{C}_v^k}(\lambda x_1 + (1-\lambda)x_2) \preceq \lambda \mathbf{F}_{\mathbf{C}_v^k}(x_1) \oplus (1-\lambda)\mathbf{F}_{\mathbf{C}_v^k}(x_2). \tag{3}$$

From (2), one may observe that $\mathbf{F}_{\mathbf{C}_v^k}$ is convex with respect to $\preceq$ means

$$f_{c(t)}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f_{c(t)}(x_1) + (1-\lambda) f_{c(t)}(x_2),$$

for all $t \in [0,1]^k$; $t$ is same in both sides. So we can conclude that $\mathbf{F}_{\mathbf{C}_v^k}$ is convex with respect to $\preceq$ if and only if $f_{c(t)}(x)$ is a convex function on $D$ for every $t \in [0,1]^k$.

Now, the differentiability and non-differentiability for interval valued functions in parametric form are stated.

**Definition 4** The interval valued function, $\mathbf{F}_{\mathbf{C}_v^k} : R^n \to I(R)$ is said to be differentiable at $x^*$ if $f_{c(t)}$ is differentiable at $x^*$ for every $t \in [0,1]^k$. If for at least one $t \in [0,1]^k$, $f_{c(t)}$ is not differentiable at $x^*$, then $\mathbf{F}_{\mathbf{C}_v^k}$ is called non-differentiable function at $x^*$.

*Example 1* An interval valued function, $\mathbf{F}_{\mathbf{C}_v^2} : R^2 \to I(R)$ define as,

$$\begin{aligned}\mathbf{F}_{\mathbf{C}_v^2}(x_1,x_2) &= [1,3]|x_1| \oplus [-2,1]\,(|x_1| - |x_2|) \\ &= \left\{ f_{c(t)}(x_1,x_2) | c(t) \in \mathbf{C}_v^2,\ f_{c(t)} : R^2 \to R \right\},\end{aligned}$$

where $f_{c(t)}(x_1,x_2) = (1+2t_1)|x_1| \oplus (-2+3t_2)(|x_1|-|x_2|)$. Since $f_{c(t)}$ is non-differentiable function at $(0,0)$ for each $t_1, t_2$, so $\mathbf{F}_{\mathbf{C}_v^2}$ is non-differentiable interval valued function at $(0,0)$.

Let $X$ be a locally convex real vector space with dual space $X'$; let $C$ be an open convex subset of $X$; let $\xi \in X'$; let $f : X \to R$.

**Definition 5** [3] The normal cone to the set $C$ at $x^* \in X$ denoted by $N_C(x^*)$, is defined as

$$N_C(x^*) = \left\{ \xi \in X' |\ (x-x^*)^T \xi \leq 0, \forall x \in X \right\}.$$

**Definition 6** [3] At a point $x^* \in X$, $\xi \in X'$ is said to be the sub-gradient of a convex function $f$ if

$$(x-x^*)^T \xi \leq f(x) - f(x^*), \forall x \in X.$$

**Definition 7** [3] At a point $x^* \in X$, $\xi \in X'$ is said to be the sub-gradient of a strictly convex function $f$ if

$$(x-x^*)^T \xi < f(x) - f(x^*), \forall x \in X.$$

**Definition 8** [3] The set of all sub-gradients of $f$ at $x^*$ is called the sub-differential of $f$ at $x^*$ and is denoted by $\partial f(x^*)$. That means

$$\partial f(x^*) = \left\{ \xi \in X' \mid (x-x^*)^T \xi \leq f(x) - f(x^*), \forall x \in X \right\}.$$

**Proposition 1** *[8] Let $f$ be an m-dimensional convex vector functions on the convex set $\Gamma \subset R^n$. Then either*

**(I)** $f(x) <_v 0$ *has a solution* $x \in \Gamma$ *or*
**(II)** $p^T f(x) \geq 0$ *for all* $x \in \Gamma$ *for some* $p \geqq_v 0$, $p \in R^m$ *but never both.*

## 3 Interval optimization problem

Consider the following interval optimization problem as,

$$(IOP) \quad \min \ \mathbf{F}_{\mathbf{C}_v^k}(x)$$
$$\text{subject to } \mathbf{G}^j_{D_v^{m_j}}(x) \preceq \mathbf{0}, \ \ j \in \Lambda_p,$$

where $\mathbf{F}_{\mathbf{C}_v^k}, \mathbf{G}^j_{D_v^{m_j}} : R^n \rightarrow I(R)$ are represented by the sets, $\mathbf{F}_{\mathbf{C}_v^k}(x) = \{f_{c(t)}(x) \mid f_{c(t)} : R^n \rightarrow R, \ c(t) \in \mathbf{C}_v^k\}$ and $\mathbf{G}^j_{D_v^{m_j}}(x) = \{g^j_{d(t'_j)}(x) \mid g^j_{d(t'_j)}(x) : R^n \rightarrow R, d(t'_j) \in \mathbf{D}_v^{m_j}\}$.

Following the partial orderings as in expression (2), the feasible region of $IOP$ can be expressed as the set,

$$\mathcal{F} = \{x \in R^n : \mathbf{G}^j_{D_v^{m_j}}(x) \preceq \mathbf{0}, \ j \in \Lambda_p\}$$
$$= \{x \in R^n : g^j_{d(t'_j)}(x) \leq 0, \ \forall \, t'_j, \ j \in \Lambda_p\} \equiv \{x \in R^n : g^R_j(x) \leq 0, \ j \in \Lambda_p\},$$

where $g^R_j(x) = \max_{t'_j \in [0,1]^{m_j}} g^j_{d(t'_j)}(x), \ \ j \in \Lambda_p$.

For weight function $w(t)$ define as $w : [0, 1]^k \rightarrow R_+$, we construct a deterministic optimization problem as follows.

$$(IOP_w) \quad \min_{x \in \mathcal{F}} \int_k w(t) f_{c(t)}(x) \, dt,$$

where $\int_k = \underbrace{\int_0^1 \int_0^1 \ldots \int_0^1}_{\text{k times}}$, $t = (t_1, t_2, \ldots, t_k)^T$ and $dt = dt_1 dt_2 \ldots dt_k$.

Here $t_1, t_2, \ldots, t_k$ are mutually independent and each $t_i$ varies in [0, 1]. So the objective function of $IOP_w$ is a function of $x$ only, say $\Phi(x)$. Hence $IOP_w$ becomes

$$\min_{x \in \mathcal{F}} \Phi(x), \quad \text{where } \Phi(x) = \int_k w(t) f_{c(t)}(x) \, dt.$$

This is a general nonlinear optimization problem, which is free from interval uncertainty.

**Definition 9** [4] Any point $x^* \in \mathcal{F}$ is called an efficient solution of $IOP$ if there is no $x \in \mathcal{F}$ with

$$f_{c(t)}(x) \leq f_{c(t)}(x^*), \ \forall\, t \in [0,1]^k \text{ and for at least one } \hat{t} \neq t \ f_{c(\hat{t})}(x) < f_{c(\hat{t})}(x^*). \tag{4}$$

The problem, $IOP$ is said to be a convex if the objective and constraint functions are interval valued convex functions with respect to $\preceq$.

**Theorem 1** *[4] If $IOP$ is an interval valued convex programming problem then $IOP_w$ is a convex programming problem.*

The relation between the solution of $IOP$ and $IOP_w$ is developed as follows.

**Theorem 2** *Any point $x^* \in \mathcal{F}$ is an efficient solution of the convex $IOP$ if and only if $x^*$ is an optimal solution of $IOP_w$.*

*Proof* Let $x^*$ be an efficient solution of a convex $IOP$, then there exists no $x \in \mathcal{F}$ satisfying the relation (4). Since $IOP$ is a convex interval optimization problem, so the objective function $\mathbf{F}_{\mathbf{C}_v^k}$ is interval valued convex functions with respect to $\preceq$. Therefore $f_{c(t)}$ is convex on a convex set $\mathcal{F}$ for each $t$. From relation (4), this means that the following system has no solution on $\mathcal{F}$.

$$f_{c(t)}(x) - f_{c(t)}(x^*) \leq 0, \ \forall\, t \in [0,1]^k \text{ and for at least one } \hat{t} \neq t \ f_{c(\hat{t})}(x) - f_{c(\hat{t})}(x^*) < 0. \tag{5}$$

Assume $F(x) = (f_{c(t)}(x) - f_{c(t)}(x^*), f_{c(\hat{t})}(x) - f_{c(\hat{t})}(x^*))^T$. The system of inequalities, (5) can be write as $F(x) \leqq_v 0$ has no solution, implies that $F(x) <_v 0$ has no solution. Hence from the Proposition 1, there exists $W = (w(t), w(\hat{t}))^T \neq 0$, where $w : [0,1]^k \to R_+$ such that $W^T F(x) \geq 0$ for all $x \in \mathcal{F}$ has solution. This is equivalent to

$$w(t)(f_{c(t)}(x) - f_{c(t)}(x^*)) + w(\hat{t})(f_{c(\hat{t})}(x) - f_{c(\hat{t})}(x^*)) \geq 0. \tag{6}$$

Integrating with respect to $t = (t_1, t_2, \ldots, t_k)$ and $\hat{t} = (\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k)$, relation (6) is equivalent to

$$\int_k w(t) f_{c(t)}(x)dt \geq \int_k w(t) f_{c(t)}(x^*)dt \ \ \forall\, x \in \mathcal{F}.$$

This implies $\Phi(x) \geq \Phi(x^*)$. Hence $x^*$ is an optimal solution of $IOP_w$.

Conversely, let $x^* \in \mathcal{F}$ be an optimal solution of $IOP_w$. Assume that $x^*$ is not an efficient solution of $IOP$, then there is some $x \in \mathcal{F}$ with

$$f_{c(t)}(x) \leq f_{c(t)}(x^*), \ \forall\, t \in [0,1]^k \text{ and for at least one } \hat{t} \neq t, \ f_{c(\hat{t})}(x) < f_{c(\hat{t})}(x^*).$$

For real valued functions $w : [0, 1]^k \to R_+$, the above relations implies that

$$\int_k w(t) f_{c(t)}(x)dt < \int_k w(t) f_{c(t)}(x^*)dt \equiv \Phi(x) < \Phi(x^*).$$

This is the contradiction that $x^*$ is an optimal solution of $IOP_w$. Hence $x^*$ is an efficient solution of $IOP$. □

## 4 Optimality conditions for $IOP$

Consider the following deterministic optimization problem,

$$(P) \quad \min f(x) \ \text{ subject to } g_i(x) \leq 0,\ i \in \Lambda_p,\ x \in X,$$

where $f, g_i : X \to R$ are continuous convex real valued functions, and $X$ is a convex subset of $R^n$.

**Theorem 3** [11](Fritz John necessary optimality). *If $x^*$ is an optimal solution of the problem $P$, and for some $x \in X$, $g_i(x) < 0, \forall i \in \Lambda_p$, then there exist $\xi^* \geq 0$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_p^*)^T \geqq_v 0$, $\lambda^* \neq 0$ such that*

$$\lambda_i^* g_i(x^*) = 0,\ i \in \Lambda_p,$$
$$0 \in \xi^* \partial f(x^*) + \sum_{i \in \Lambda_p} \lambda_i^* \partial g_i(x^*) + N_X(x^*).$$

Fritz John necessary optimality conditions for non-differentiable convex $IOP$ have been derived as follows.

**Theorem 4** *If $x^*$ is an efficient solution of the convex $IOP$ and for some $x \in \mathcal{F}$, $g_j^R(x) < 0$, $j \in \Lambda_p$, then there exist $\xi \geq 0$ and $\lambda^R = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_p^R)^T \geqq_v 0$, $\lambda^R \neq 0$ such that*

$$\lambda_j^R g_j^R(x^*) = 0,\ j \in \Lambda_p,$$
$$0 \in \xi \partial \Phi(x^*) + \sum_{j \in \Lambda_p} \lambda_j^R \partial g_j^R(x^*) + N_{\mathcal{F}}(x^*).$$

*Proof* Let $x^*$ be an efficient solution of convex $IOP$. From Theorem 2, $x^*$ is an optimal solution of convex problem $IOP_w$ and for some $x \in \mathcal{F}$, $g_j^R(x) < 0$, $j \in \Lambda_p$. Hence from Theorem 3, there exist $\xi \geq 0$ and $\lambda^R = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_p^R)^T \geqq_v 0$, $\lambda^R \neq 0$ such that

$$\lambda_j^R g_j^R(x^*) = 0,\ j \in \Lambda_p,$$
$$0 \in \xi \partial \Phi(x^*) + \sum_{j \in \Lambda_p} \lambda_j^R \partial g_j^R(x^*) + N_{\mathcal{F}}(x^*).$$

This completes the proof. □

**Theorem 5** *Suppose* $\mathbf{F}_{\mathbf{C}_v^k}$ *and* $\mathbf{G}^j_{D_v^{m_j}}$, $j \in \Lambda_p$ *are interval valued convex functions with respect to* $\preceq$. *There exist* $x^* \in \digamma$ *and* $\xi > 0$, $\lambda^R = (\lambda_1^R, \lambda_2^R, \ldots, \lambda_p^R)^T \geqq_v 0$, $\lambda^R \neq 0$ *such that*

$$\lambda_j^R g_j^R(x^*) = 0, \; j \in \Lambda_p,$$
$$0 \in \xi \partial \Phi(x^*) + \sum_{j \in \Lambda_p} \lambda_j^R \partial g_j^R(x^*) + N_\digamma(x^*). \tag{7}$$

*Then* $x^*$ *is an efficient solution of* $IOP$.

*Proof* From (7), it follows that there is some $\alpha \in \partial \Phi(x^*)$, $v_j^R \in \partial g_j^R(x^*)$, $j \in \lambda_p$, and $n \in N_\digamma(x^*)$ such that $0 = \xi\alpha + \sum_{j \in \Lambda_p} \lambda_j^R v_j^R + n$. This implies

$$0 = (x - x^*)^T \left( \xi\alpha + \sum_{j \in \Lambda_p} \lambda_j^R v_j^R + n \right). \tag{8}$$

If $x^*$ is not an efficient solution of $IOP$, then there exists some $x \in \digamma$ such that

$$f_{c(t)}(x) \le f_{c(t)}(x^*), \; \forall\, t \in [0,1]^k \text{ and for at least one } \hat{t} \neq t \;\; f_{c(\hat{t})}(x) < f_{c(\hat{t})}(x^*).$$

Since $\mathbf{F}_{\mathbf{C}_v^k}$ is an interval valued convex function with respect to $\preceq$, so for each $t$, $f_{c(t)}$ is a real valued convex function. The convexity of $f_{c(t)}(x)$ implies the convexity of $\Phi(x)$, we have $(x - x^*)^T \alpha < 0$. For $\xi > 0$, this relation implies

$$(x - x^*)^T \xi\alpha < 0. \tag{9}$$

From $\lambda_j^R \ge 0$, $g_j^R(x) \le 0$, $\lambda_j^R g_j^R(x^*) = 0$, $j \in \Lambda_p$, we have $\lambda_j^R g_j^R(x) \le \lambda_j^R g_j^R(x^*)$. Since $\mathbf{G}^j_{D_v^{m_j}}$, $\forall j$ are interval valued convex functions with respect to $\preceq$, so for each $t'_j$, $g^j_{d(t'_j)}$ are convex real valued functions. Convexity of $g^j_{d(t'_j)}$ implies the convexity of $g_j^R$, we have

$$\sum_{j \in \Lambda_p} (x - x^*)^T \lambda_j^R v_j^R \le 0. \tag{10}$$

Also for $n \in N_\digamma(x^*)$, we have

$$(x - x^*)^T n \le 0. \tag{11}$$

Adding (9), (10) and (11), we obtain

$$(x - x^*)^T (\xi\alpha + \sum_{j \in \Lambda_p} \lambda_j^R v_j^R + n) < 0.$$

This contradicts (8). Hence $x^*$ is an efficient solution of $IOP$. □

## 5 Duality theory for $IOP$

The dual problem of the primal $IOP$ is defined as follows.

$$(DIOP)\quad \max\ \mathbf{F}_{\mathbf{C}_v^k}(y)$$
$$\text{subject to } 0 \in \xi\partial\Phi(y) + \sum_{j\in\Lambda_p} \lambda_j^R \partial g_j^R(y) + N_F(y), \tag{12}$$
$$\lambda_j^R g_j^R(y) \geq 0,\ \ j \in \Lambda_p.$$

Denote $\Pi$ as the feasible set of $DIOP$ and define as

$$\Pi = \left\{(y, \xi, \lambda^R) |\ 0 \in \xi\partial\Phi(y) + \sum_{j\in\Lambda_p} \lambda_j^R \partial g_j^R(y) + N_F(y), \lambda_j^R g_j^R(y) \geq 0,\ \ j \in \Lambda_p\right\}.$$

Now, we define the efficient solution of $DIOP$ as efficient solution of $IOP$.

**Definition 10** Any point $(y^*, \xi^*, \lambda^{R*}) \in \Pi$ is called an efficient solution of $DIOP$ if there is no $(y, \xi, \lambda^R) \in \Pi$ with

$$f_{c(t)}(y) \geq f_{c(t)}(y^*),\ \forall\, t \in [0, 1]^k \text{ and for at least one } \hat{t} \neq t\ f_{c(\hat{t})}(y) > f_{c(\hat{t})}(y^*).$$

**Theorem 6** (Weak Duality). *Let* $\mathbf{F}_{\mathbf{C}_v^k}$ *and* $\mathbf{G}^j_{D_v^{m_j}}$, $j \in \Lambda_p$ *are interval valued convex functions with respect to* $\preceq$. *If* $x$ *is a feasible solution of primal problem* $IOP$ *and* $(y, \xi, \lambda^R)$ *is a feasible solution of dual problem* $DIOP$. *Then* $\mathbf{F}_{\mathbf{C}_v^k}(x) \not\prec \mathbf{F}_{\mathbf{C}_v^k}(y)$.

*Proof* We assume that $\mathbf{F}_{\mathbf{C}_v^k}(x) \prec \mathbf{F}_{\mathbf{C}_v^k}(y)$. From (12), for some $\alpha \in \partial\Phi(y)$, $v_j^R \in \partial g_j^R(y)$, $j \in \Lambda_j$ and $n \in N_F(y)$ such that $0 = \xi\alpha + \sum_{j\in\Lambda_p} \lambda_j^R v_j^R + n$. This yields

$$0 = (x-y)^T \left(\xi\alpha + \sum_{j\in\Lambda_p} \lambda_j^R v_j^R + n\right). \tag{13}$$

Similar way to derive relation (9), we have

$$(x-y)^T \xi\alpha < 0. \tag{14}$$

By $x \in F$, $(y, \xi, \lambda^R) \in \Pi$, we have the following inequalities

$$\sum_{j\in\Lambda_p} \lambda_j^R g_j^R(y) \geq 0 \geq \sum_{j\in\Lambda_p} \lambda_j^R g_j^R(x).$$

From the convexity of $\mathbf{G}^j_{D_v^{m_j}}$, $j \in \Lambda_p$ with respect to $\preceq$, we get

$$\sum_{j\in\Lambda_p} (x-y)^T \lambda_j^R v_j^R \leq 0. \tag{15}$$

Also, since $n \in N_F(y)$,

$$(x-y)^T n \leq 0. \tag{16}$$

Combining (14), (15) and (16), we obtain

$$(x-y)^T \left( \xi\alpha + \sum_{j \in \Lambda_p} \lambda_j^R v_j^R + n \right) < 0,$$

which contradicts (13). Hence, this completes the proof. □

**Theorem 7** (Strong Duality). *Let $x^*$ be an efficient solution of $IOP$, and at $x^* \in F$, $g_j^R(x^*) < 0$, $\forall j$, then there exists $\xi^* \geq 0$ and $\lambda^{R*} = (\lambda_1^{R*}, \lambda_2^{R*}, \ldots, \lambda_p^{R*})^T \geqq_v 0$, such that $(x^*, \xi^*, \lambda^{R*}) \in \Pi$. If $\mathbf{F}_{\mathbf{C}_v^k}$ and $\mathbf{G}^j_{D_v^{m_j}}$, $\forall j$ are interval valued convex functions with respect to $\preceq$ then $(x^*, \xi^*, \lambda^{R*})$ is an efficient solution of $DIOP$.*

*Proof* Since $x^*$ is an efficient solution of $IOP$, and at $x^* \in F$, $g_j^R(x^*) < 0$, $j \in \Lambda_p$. From Theorem 4, there exists $\xi^* \geq 0$ and $\lambda^{R*} \geq 0$ such that

$$\lambda_j^{R*} g_j^R(x^*) = 0,\ j \in \Lambda_p,$$
$$0 \in \xi^* \partial \Phi(x^*) + \sum_{j \in \Lambda_p} \lambda_j^{R*} \partial g_j^R(x^*) + N_F(x^*).$$

Hence $(x^*, \xi^*, \lambda^{R*}) \in \Pi$. Next suppose that $(x^*, \xi^*, \lambda^{R*})$ is not an efficient solution of problem $DIOP$. Then, there exists a feasible solution $(x, \xi, \lambda^R)$ of $DIOP$ such that

$$f_{c(t)}(x) \geq f_{c(t)}(x^*),\ \forall\, t \in [0,1]^k \text{ and for at least one } \hat{t} \neq t,\ f_{c(\hat{t})}(x) > f_{c(\hat{t})}(x^*).$$

That is, $\mathbf{F}_{\mathbf{C}_v^k}(x) \succ \mathbf{F}_{\mathbf{C}_v^k}(x^*)$, which contradicts the Theorem 6. Hence $(x^*, \xi^*, \lambda^{R*})$ is an efficient solution of $DIOP$. □

**Theorem 8** (Converse Duality). *Let $x^*$ and $(y^*, \xi^*, \lambda^{R*})$ be feasible solution for primal problem $IOP$ and dual problem $DIOP$, respectively. For all feasible point $(x, y)$, $\xi^*\Phi + \sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R$ is strictly convex at $y^*$ and $\Phi(x^*) \leq \Phi(y^*)$. Then $x^* = y^*$.*

*Proof* Assume that $x^* \neq y^*$. Since $(y^*, \xi^*, \lambda^{R*})$ is a feasible solution of dual problem $DIOP$, so from (12), for some $\alpha^* \in \partial\Phi(y^*)$, $v_j^{R*} \in \partial g_j^R(y^*)$, $j \in \Lambda_j$ and $n \in N_F(y^*)$ such that $0 = \xi^*\alpha + \sum_{j \in \Lambda_p} \lambda_j^{R*} v_j^R + n$. This yields

$$0 = (x^* - y^*)^T \left( \xi^*\alpha + \sum_{j \in \Lambda_p} \lambda_j^{R*} v_j^R + n \right).$$

Since $n \in N_F(y^*)$, $(x^* - y^*)^T n \le 0$, we have

$$(x^* - y^*)^T \left( \xi^* \alpha + \sum_{j \in \Lambda_p} \lambda_j^{R*} v_j^R \right) \ge 0. \tag{17}$$

From the strictly convexity of $\xi^* \Phi + \sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R$, (17) implies

$$\xi^* \Phi(x^*) + \sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R(x^*) > \xi^* \Phi(y^*) + \sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R(y^*). \tag{18}$$

Since $x^* \in F$ and $(y^*, \xi^*, \lambda^{R*}) \in \Pi$, so $\sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R(x^*) \le 0$ and $\sum_{j \in \Lambda_p} \lambda_j^{R*} g_j^R(y^*) \ge 0$. From (18), we have $\xi^* \Phi(x^*) > \xi^* \Phi(y^*)$, which contradicts the assumption $\Phi(x^*) \le \Phi(y^*)$. Therefore $x^* = y^*$. □

*Example 2* Consider the following interval optimization problem

$$\begin{aligned} (IOP) \qquad & \min \ \mathbf{F}(x_1, x_2) = [|x_1 - x_2|, |x_1 - x_2| + 2] \\ & \text{subject to } g(x_1, x_2) = |x_1 - 1| \le 0, \\ & (x_1, x_2) \in X, \end{aligned}$$

where $X = \{x_1, x_2 | \, |x_1| \le 1, |x_2| \le 1\}$.

For weight function $w : [0, 1] \to R$, the corresponding deterministic problem $IOP_w$ is

$$\begin{aligned} (IOP_w) \qquad & \min \ \Phi(x_1, x_2) = \int_0^1 w(t)(|x_1 - x_2| + 2t)dt \\ & \text{subject to } g(x_1, x_2) = |x_1 - 1| \le 0, \\ & (x_1, x_2) \in X. \end{aligned}$$

In particular $w(t) = 1$, the problem $IOP_w$ becomes,

$$\begin{aligned} (IOP_w) \qquad & \min \ \Phi(x_1, x_2) = |x_1 - x_2| + 1 \\ & \text{subject to } g(x_1, x_2) = |x_1 - 1| \le 0, \\ & (x_1, x_2) \in X. \end{aligned}$$

Since $(x_1^*, x_2^*) = (1, 1)$ is the minimum solution of $IOP_w$, so from Theorem 2, $(x_1^*, x_2^*) = (1, 1)$ is an efficient solution of $IOP$.

We have,

$$\begin{aligned} \partial \Phi(1, 1) &= \{(1, 1)\}, \\ \partial g(1, 1) &= \{(x_1, x_2) | x_2 \ge 0, x_1 \ge -1\}, \\ N_X(1, 1) &= \{(x_1, x_2) | x_2 \ge 0, x_1 \ge 0\}. \end{aligned}$$

Then, there exist $\xi = 0, \lambda_1 = 1$ such that

$$(0, 0) \in \xi \partial \Phi(1, 1) + \lambda_1 \partial g(1, 1) + N_X(1, 1) \text{ and } \lambda_1 g(1, 1) = 0.$$

Theorem 4 is verified in this example.

## 6 Conclusion

The existence of the solution of interval optimization problem is discussed, where the objective as well as the constraint functions are non-differentiable interval valued. Necessary and sufficient optimality conditions for this problem are derived. Further, a dual problem for this objective is defined and developed the relation between the primal and the dual interval optimization problems. The duality theory and optimality conditions for a general multi-objective interval optimization problem is established without assuming the differentiability of the objective and constraint functions.

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